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Insensitizing controls for the Cahn–Hilliard type equation

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Abstract. This paper is addressed to showing the existence of insensitizing controls for the one-dimensional Cahn–Hilliard type equation with a superlinear nonlinearity. We solve this problem by reducing the original problem to a controllability problem. The crucial point in this paper is an observability estimate for a linearized cascade system of the Cahn–Hilliard type equation. In order to obtain this observability estimate, we establish a global Carleman estimate for a fourth order parabolic operator.

Keywords: Cahn–Hilliard type equation, insensitizing control, controllability, observability, superlinear nonlinearity, global Carleman estimate.

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1 Introduction

Set $I = (0,1)$, $T > 0$, and $Q = I \times (0,T)$. Let ω and $\mathcal{O}$ be nonempty open subsets of I. We consider the Cahn–Hilliard type equation posed on the finite interval I satisfying some homogeneous boundary conditions and an initial condition, namely

$$\begin{cases} y_t + y_{xxxx} + f(y) = \xi + h\chi_\omega & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = y^0(x) + \tau z^0(x) & \text{in } I, \end{cases} \tag{1.1}$$

where f is a C^1 function defined on R verifying $f'' \in L^\infty_{loc}(R)$, $f(0) = 0$ and

$$\lim_{|s|\to\infty} \frac{f'(s)}{\log(1+ \mid s \mid)} = 0, \tag{1.2}$$

$\xi \in L^2(Q)$ and $y^0 \in L^2(I)$ are given, $z^0 \in L^2(I)$ is unknown with $\|z^0\|_{L^2(I)} = 1$, τ is a small unknown real number, and $h \in L^2(Q)$ is a control function to be determined. Here χ_ω denotes the characteristic function of ω.

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The Cahn–Hilliard equation is an equation of mathematical physics which describes the process of phase separation, by which the two components of a binary fluid spontaneously separate and form domains pure in each component. It arises as a phenomenological model for isothermal phase separation in a binary alloy, see Cahn [7, 8] and Hilliard [20] for a derivation, [15, 23, 27] for general analysis, and the reviews given in [16].

Let us define

$$\Phi(y(\cdot,\cdot,\tau,h)) = \frac{1}{2}\int_0^T\int_{\mathcal{O}} \mid y(x,t,\tau,h) \mid^2 dx\,dt,$$

where $y(\cdot,\cdot,\tau,h)$ is the solution of (1.1) associated to τ.

The following control problem is addressed: Does there exist a control $h \in L^2(Q)$ such that

$$\frac{d\Phi(y(\cdot,\cdot,\tau,h))}{d\tau}\bigg|_{\tau=0} = 0$$

holds? The problem is interesting, and attracts many authors' attention. We call it insensitizing control problem. Next, we investigate the existence of insensitizing controls for Φ about the system (1.1), their definitions are as follows.

Definition 1.1. The control h is said to insensitize the functional Φ if for every z^0 satisfying $\|z^0\|_{L^2(I)} = 1$, the corresponding solution y of (1.1) satisfies

$$\frac{d}{d\tau}\Phi(y(\cdot,\cdot,\tau,h))\bigg|_{\tau=0} = 0.$$

The insensitizing control problem consists in finding a control function such that some functional of the state is locally insensitive to the perturbations of these initial and boundary data. The concept of insensitizing control was introduced by J. L. Lions [21]. Later on, Bodart and Fabre proposed the weakened notion of $\varepsilon-$insensitizing control in [2]. A similar result was proved by Teresa [12] in unbounded domains. The first results on the existence and non-existence of insensitizing controls were proved in [13]. For more general nonlinearities, see [3, 4, 5]. A similar result for wave equations was obtained in [11, 26].

The main purpose in our paper is to study the existence of insensitizing controls for the Cahn–Hilliard equation. As far as we know, there is no insensitivity result for this equation. In this sense, this is the first attempt to consider insensitizing controls problem for the Cahn–Hilliard equation. In order to solve this problem, we establish a new observability estimate (see Theorem 1.2).

Following the methods introduced in [21] and developed in [2, 11, 13, 26], one gets that the existence of a control h insensitizing the functional Φ along the solutions of (1.1) is equivalent to the existence of a control h such that the solution $(\overline{y}, q)$ of the cascade system (1.3)–(1.4)

$$\begin{cases} \overline{y}_t + \overline{y}_{xxxx} + f(\overline{y}) = \xi + h\chi_\omega & \text{in } Q, \\ \overline{y}(0,t) = 0 = \overline{y}(1,t) & \text{in } (0,T), \\ \overline{y}_x(0,t) = 0 = \overline{y}_x(1,t) & \text{in } (0,T), \\ \overline{y}(x,0) = y^0(x) & \text{in } I \end{cases} \tag{1.3}$$

$$\begin{cases} -q_t + q_{xxxx} + f'(\overline{y})q = \overline{y}\chi_{\mathcal{O}} & \text{in } Q, \\ q(0,t) = 0 = q(1,t) & \text{in } (0,T), \\ q_x(0,t) = 0 = q_x(1,t) & \text{in } (0,T), \\ q(x,T) = 0 & \text{in } I \end{cases} \tag{1.4}$$

satisfies

$$q(x,0) = 0.$$

Namely, system (1.3)–(1.4) is null controllable. The null controllability has been widely investigated for the heat equation and there has been a great number of results (see for instance [6, 22] and the references therein for a detailed account). To our best knowledge, there have been limited publications on the controllability of higher order parabolic equations. Among them, Díaz [14] considered the approximate controllability and non-approximate controllability of higher order parabolic equations. The null boundary controllability for a one-dimensional fourth order parabolic equation was studied in [6, 10]. Cerpa [9] considered the local boundary controllability for an especial one-dimensional fourth order parabolic equation (Kuramoto–Sivashinsky equation). Recently, Zhou [28] considered the null controllability for one-dimensional semilinear fourth order parabolic equations.

In order to investigate system (1.3)–(1.4), we firstly consider the linearized system of (1.3)–(1.4)

$$\begin{cases} y_t + y_{xxxx} + ay = \xi + h\chi_\omega & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = y^0(x) & \text{in } I \end{cases} \tag{1.5}$$

$$\begin{cases} -q_t + q_{xxxx} + bq = y\chi_{\mathcal{O}} & \text{in } Q, \\ q(0,t) = 0 = q(1,t) & \text{in } (0,T), \\ q_x(0,t) = 0 = q_x(1,t) & \text{in } (0,T), \\ q(x,T) = 0 & \text{in } I \end{cases} \tag{1.6}$$

where $a, b \in L^\infty(Q)$.

The adjoint system of (1.5)–(1.6) is

$$\begin{cases} p_t + p_{xxxx} + b(x,t)p = 0 & \text{in } Q, \\ p(0,t) = 0 = p(1,t) & \text{in } (0,T), \\ p_x(0,t) = 0 = p_x(1,t) & \text{in } (0,T), \\ p(x,0) = p^0(x) & \text{in } I \end{cases} \tag{1.7}$$

$$\begin{cases} -z_t + z_{xxxx} + a(x,t)z = p\chi_{\mathcal{O}} & \text{in } Q, \\ z(0,t) = 0 = z(1,t) & \text{in } (0,T), \\ z_x(0,t) = 0 = z_x(1,t) & \text{in } (0,T), \\ z(x,T) = 0 & \text{in } I. \end{cases} \tag{1.8}$$

According to the duality argument, the observability estimate of (1.7)–(1.8) is important for the insensitizing control problem.

Theorem 1.2. *For every $p^0 \in L^2(I)$, if (p,z) is the solution to (1.7)–(1.8), there exist $M > 0$ and $C(T) > 0$, such that*

$$\int_Q e^{-\frac{M}{t}} z^2 \, dx \, dt \leq e^{C(T)\left(\|a\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)} + 1\right)} \int_{\omega\times(0,T)} z^2 \, dx \, dt. \tag{1.9}$$

The following duality identity for the solutions of (1.5)–(1.6) and (1.7)–(1.8) holds

$$\int_Q (\xi + h\chi_\omega) z \, dx \, dt = \int_I \left(q(x,0) p^0(x) - z(x,0) y^0(x) \right) dx \tag{1.10}$$

for every $h \in L^2(Q)$, $y^0 \in L^2(\Omega)$, $p^0 \in L^2(\Omega)$. Indeed, multiplying (1.5) by z and integrating by parts in account of the boundary and initial (final) conditions in (1.5)–(1.6) and (1.7)–(1.8), we can get (1.10).

By the observability estimate (1.9) of the linearized system (1.5)–(1.6) and the fixed point theorem, we have the following result:

Theorem 1.3. *Let $\mathcal{O}$ and ω be nonempty open subsets of I satisfying $\omega \cap \mathcal{O} \neq \emptyset$ and $y^0 = 0$. Then for any $\xi \in L^2(Q)$ verifying $e^{\frac{M}{2t}} \xi \in L^2(Q)$, one can find a control function $h \in L^2(Q)$ insensitizing the functional Φ along the solution of* (1.1)*, where M is same as in Theorem 1.2.*

Remark 1.4. In view of Theorem 1.3, we can obtain the null controllability of (1.3)–(1.4) with the nonlinearities $f(s) = o(s(\log(|s|)))$ for $|s| \to \infty$. For the scalar Cahn–Hilliard type equation

$$\begin{cases} y_t + y_{xxxx} + F(y) = h\chi_\omega & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = y^0(x) & \text{in } I \end{cases} \tag{1.11}$$

with nonlinearities such that $F(s) = o(s(\log^{\frac{7}{2}}(|s|)))$ for $|s| \to \infty$, it seems possible to obtain the null controllability of (1.11). Indeed, following the same idea as in [18], we can choose a small time $T^* < T$ and find a control h that drives the solution to zero at T^*, then extend h by zero to the rest interval $[T^*, T]$.

However, for the system (1.3)–(1.4), since the existence of the nonhomogeneous term ξ, the above method does not work. More precisely, even though we can obtain the null controllability of (1.3)–(1.4) at a small time T^*, the zero control in $[T^*, T]$ cannot guarantee the null controllability of (1.3)–(1.4) at T owing to ξ. According to the existent methods, the best result for the nonlinearities in (1.3)–(1.4) we can obtain is $f(s) = o(s(\log(|s|)))$ for $|s| \to \infty$. The key point is the estimate (4.7) in Section 4. The same reason can also be found in [3] which considers the insensitizing controls for a heat equation.

The paper is organized as follows. In Section 2, we present some well-posedness results by the classical semigroup theory, multipliers method and suitable energy estimates. Then, we establish a Carleman estimate for the fourth order parabolic operator. The observability estimate is established in Section 3. In Section 4, by means of the variational approach, the observability estimate in the above section and Kakutani's fixed point theorem, we establish the existence of insensitizing controls for the Cahn–Hilliard equation.

2 Some preliminaries

In order to prove Theorem 1.2, we should establish a global Carleman estimate for a fourth order parabolic operator.

Let $\psi \in C^\infty(\overline{\Omega})$ satisfy that $\psi > 0$ in Ω, $\psi(0) = \psi(1) = 0$, $\|\psi\|_{C(\overline{\Omega})} = 1$, $|\ \psi_x\ | > 0$ in $\overline{\Omega} \backslash \omega_0$, $\psi_x(0) > 0$ and $\psi_x(1) < 0$. For any given positive constants λ and μ, we set $d(x,t) = \frac{e^{\mu(\psi(x)+3)} - e^{5\mu}}{t(T-t)}$,

$\theta(x,t) = e^{\lambda d(x,t)}$ and $\varphi(x,t) = \frac{e^{\mu(\psi(x)+3)}}{t(T-t)}$, $\forall (x,t) \in Q$. Let P be an operator

$$Py := y_t + y_{xxxx},$$

defined on $\mathcal{U} := \{y \in L^2(0,T;H^4(I)) \mid y(t,0) = y(t,1) = y_x(t,0) = y_x(t,1) = 0,\ t \in (0,T),\ Py \in L^2(0,T;L^2(I))\}$.

Proposition 2.1. *There exist four constants $\mu_0 > 1$, $C_0 > 0$, $C_1 > 0$ and $C_2 > 0$ such that for $\mu = \mu_0$ and for every $\lambda \geq C_0(T+T^2)$ and $y \in \mathcal{U}$, we have*

$$\begin{aligned}&\int_Q \Big(\frac{1}{\lambda\varphi}\theta^2(y_t^2 + y_{xxxx}^2) + \lambda\varphi\theta^2 y_{xxx}^2 + \lambda^3\varphi^3\theta^2 y_{xx}^2 + \lambda^5\varphi^5\theta^2 y_x^2 + \lambda^7\varphi^7\theta^2 y^2\Big)dx\,dt \\ &\quad \leq C_1\left(\int_Q \theta^2|Py|^2\,dx\,dt + \int_{Q^\omega} \lambda^7\varphi^7\theta^2 y^2\,dx\,dt\right).\end{aligned} \tag{2.1}$$

Moreover,

$$\begin{aligned}&\int_Q \Big(\lambda^{-1}t(T-t)\theta^2(y_t^2 + y_{xxxx}^2) + \lambda t^{-1}(T-t)^{-1}\theta^2 y_{xxx}^2 + \lambda^3 t^{-3}(T-t)^{-3}\theta^2 y_{xx}^2 \\ &\qquad + \lambda^5 t^{-5}(T-t)^{-5}\theta^2 y_x^2 + \lambda^7 t^{-7}(T-t)^{-7}\theta^2 y^2\Big)dx\,dt \\ &\quad \leq C_2\left(\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}y^2 dxdt + \int_Q \theta^2|Py|^2\,dx\,dt\right).\end{aligned} \tag{2.2}$$

Remark 2.2. A Carleman estimate for the fourth order parabolic operator was previously obtained in [28]. Our Carleman estimate is a generalization to the result in [28]. More precisely, we can also obtain the estimate for $\int_Q \big(\frac{1}{\lambda\varphi}\theta^2(y_t^2 + y_{xxxx}^2) + \lambda\varphi\theta^2 y_{xxx}^2\big)dx\,dt$. We only sketch the proof in the Appendix.

Now, we present a regularity result for the following system

$$\begin{cases} y_t + y_{xxxx} + ay = g & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = y^0(x) & \text{in } I. \end{cases} \tag{2.3}$$

Proposition 2.3.

(i) If $g \in L^2(0,T;L^2(I))$, $a \in L^\infty(Q)$ and $y^0 \in L^2(I)$, system (2.3) has a unique mild solution y in $C([0,T];L^2(I)) \cap L^2(0,T;H_0^2(I))$. Moreover, there exists a constant $C = C(T)$, such that

$$\|y\|_{C([0,T];L^2(I))\cap L^2(0,T;H_0^2(I))} \leq Ce^{C(\|a\|_{L^\infty(Q)}+1)}\left(\|g\|_{L^2(0,T;L^2(I))} + \|y^0\|_{L^2(I)}\right).$$

(ii) If $g \in L^2(0,T;L^2(I))$, $a \in L^\infty(Q)$ and $y^0 \in H_0^2(I)$, system (2.3) has a unique mild solution y in $C([0,T];H_0^2(I)) \cap L^2(0,T;H^4(I))$. Moreover, there exists a constant $C = C(T)$, such that

$$\|y\|_{C([0,T];H_0^2(I))\cap L^2(0,T;H^4(I))} \leq Ce^{C(\|a\|_{L^\infty(Q)}+1)}\left(\|g\|_{L^2(0,T;L^2(I))} + \|y^0\|_{H_0^2(I)}\right).$$

Remark 2.4. By the classical semigroup theory, multipliers method and suitable energy estimates [19, 24], we can obtain Proposition 2.3.

3 Proof of Theorem 1.2

Applying the classical estimates for the parabolic equation to the system (1.7)–(1.8), we can obtain the following lemma.

Lemma 3.1. *System (1.7)–(1.8) has the following energy estimates*

(i)
$$\int_I p^2(t_2)\,dx \le e^{2\|b\|_{L^\infty(Q)}(t_2-t_1)} \int_I p^2(t_1)\,dx, \quad \forall t_1 < t_2;$$

(ii)
$$\|z(t)\|^2_{L^2(I)} \le \int_t^T e^{(2\|a\|_{L^\infty(Q)}+1)(s-t)} \|p(s)\|_{L^2(\mathcal{O})} ds, \quad \forall t \in [0,T].$$

In particular, we have

$$\int_I p^2\left(t+\frac{T}{4}\right) dx \le e^{\|b\|_{L^\infty(Q)}\frac{T}{2}} \int_I p^2(t)\,dx, \quad \forall t \in \left[\frac{T}{4}, \frac{3T}{4}\right],$$

and hence,
$$\int_{I\times(\frac{T}{2},T)} p^2\,dx\,dt \le e^{\|b\|_{L^\infty(Q)}\frac{T}{2}} \int_{I\times(\frac{T}{4},\frac{3T}{4})} p^2\,dx\,dt. \tag{3.1}$$

On the other hand,

$$\int_t^T \|z(s)\|^2_{L^2(I)} ds \le (T-t) e^{(2\|a\|_{L^\infty(Q)}+1)\frac{T}{2}} \int_t^T \|p(s)\|_{L^2(\mathcal{O})} ds, \quad \forall t \in \left[\frac{T}{2}, T\right],$$

thus
$$\int_{I\times(\frac{T}{2},T)} z^2 dxds \le e^{(\|a\|_{L^\infty(Q)}+1)T} \int_{\mathcal{O}\times(\frac{T}{2},T)} p^2 dxds. \tag{3.2}$$

By the same method as in [5, Lemma 2.4], a simple calculation yields

Lemma 3.2. *Set* $m_0 = \min\limits_{x\in\overline{\Omega}} \left(e^{5\mu} - e^{\mu(\psi(x)+3)}\right)$ *and* $M_0 = \max\limits_{x\in\overline{\Omega}} \left(e^{5\mu} - e^{\mu(\psi(x)+3)}\right)$.

(i) When $\lambda > \frac{7T^2}{2M_0}$, *the function* $e^{-\frac{2\lambda M_0}{T(T-t)}}(T-t)^{-7}$ *is decreasing in* $(0,T)$.

(ii) When $\lambda > \frac{15T^2}{8m_0}$, *we have* $\lambda^8\theta^2 t^{-15}(T-t)^{-15} \le 2^{30}T^{-14}m_0^{-8}e^{-8}$.

In particular, we have that for any $t \in (0, \frac{T}{2})$,

$$\begin{aligned} e^{-\frac{2\lambda M_0}{t(T-t)}} t^{-7}(T-t)^{-7} &= e^{-\frac{2\lambda M_0}{Tt}} t^{-7} \cdot e^{-\frac{2\lambda M_0}{T(T-t)}} (T-t)^{-7} \\ &\ge e^{-\frac{2\lambda M_0}{Tt}} \left(\frac{T}{2}\right)^{-7} \cdot e^{-\frac{2\lambda M_0}{T\frac{T}{2}}} \left(\frac{T}{2}\right)^{-7} \\ &= e^{-\frac{2\lambda M_0}{Tt}} \cdot C(T). \end{aligned}$$

Proof of Theorem 1.2

We first assume that $p^0 \in H_0^2(I)$.

According to Proposition 2.1, we obtain that there exists a positive λ_0, such that when $\lambda \geq \lambda_0$

$$
\begin{aligned}
&\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2 dxdt + \int_Q \lambda^5\theta^2 t^{-5}(T-t)^{-5}p_x^2\,dx\,dt \\
&\quad + \int_Q \lambda^3\theta^2 t^{-3}(T-t)^{-3}p_{xx}^2\,dx\,dt + \int_Q \lambda^1\theta^2 t^{-1}(T-t)^{-1}p_{xxx}^2\,dx\,dt \\
&\quad \leq C\left(\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt + \int_Q \theta^2|bp|^2\,dx\,dt\right) \\
&\quad \leq C\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt + \frac{1}{2}\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}z^2\,dx\,dt + \int_Q \lambda^5\theta^2 t^{-5}(T-t)^{-5}z_x^2 dxdt \\
&\quad + \int_Q \lambda^3\theta^2 t^{-3}(T-t)^{-3}z_{xx}^2\,dx\,dt + \int_Q \lambda\theta^2 t^{-1}(T-t)^{-1}z_{xxx}^2\,dx\,dt \\
&\leq C\left(\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}z^2\,dx\,dt + \int_Q \theta^2|p\chi_{\mathcal{O}} - az|^2\,dx\,dt\right) \\
&\leq C\left(\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}z^2\,dx\,dt + \frac{1}{2}\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}z^2\,dx\,dt + \int_Q \theta^2 p^2\chi_{\mathcal{O}}\,dx\,dt\right).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt + \int_Q \lambda^5\theta^2 t^{-5}(T-t)^{-5}p_x^2\,dx\,dt \\
&\quad + \int_Q \lambda^3\theta^2 t^{-3}(T-t)^{-3}p_{xx}^2\,dx\,dt + \int_Q \lambda\theta^2 t^{-1}(T-t)^{-1}p_{xxx}^2\,dx\,dt \\
&\quad \leq C\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}z^2\,dx\,dt + \int_Q \lambda^5\theta^2 t^{-5}(T-t)^{-5}z_x^2\,dx\,dt \\
&\quad + \int_Q \lambda^3\theta^2 t^{-3}(T-t)^{-3}z_{xx}^2\,dx\,dtdt + \int_Q \lambda\theta^2 t^{-1}(T-t)^{-1}z_{xxx}^2\,dx\,dt \\
&\quad \leq C\left(\int_{\omega\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}z^2\,dx\,dt + \int_Q \theta^2 p^2\chi_{\mathcal{O}}\,dx\,dt\right).
\end{aligned}
\tag{3.4}
$$

Let us consider two open sets B_1 and B_2 such that $B_1 \subset B_2 \subset \omega\cap\mathcal{O}$, and let us set $u = \lambda^7\theta^2 t^{-7}(T-t)^{-7}$. Consider a function $\xi_1 \in C_0^\infty(I)$ such that $0 \leq \xi_1 \leq 1$ in I, $\xi_1 = 1$ in B_1, $\operatorname{supp}\xi_1 \subset B_2 \subset \omega\cap\mathcal{O}$, $\frac{|\xi_{1x}|}{\xi_1^{1/2}}, \frac{|\xi_{1xx}|}{\xi_1^{1/2}}, \frac{|\xi_{1xxx}|}{\xi_1^{1/2}}, \frac{|\xi_{1xxxx}|}{\xi_1^{1/2}} \in L^\infty(Q)$.

From (3.3), we can deduce that

$$\begin{aligned}
&\int_Q \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt+\int_Q \lambda^5\theta^2 t^{-5}(T-t)^{-5}p_x^2\,dx\,dt\\
&\qquad+\int_Q \lambda^3\theta^2 t^{-3}(T-t)^{-3}p_{xx}^2\,dx\,dt+\int_Q \lambda\theta^2 t^{-1}(T-t)^{-1}p_{xxx}^2\,dx\,dt\\
&\quad\le C\int_{B_1\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt\\
&\quad\le C\int_{\mathcal{O}\times(0,T)} \lambda^7\theta^2 t^{-7}(T-t)^{-7}p^2\,dx\,dt\\
&\quad\le C\int_Q \xi_1 up\cdot p\chi_{\mathcal{O}}\,dx\,dt\\
&\quad= C\int_Q \xi_1 up\cdot(-z_t+z_{xxxx}+a(x,t)z)\,dx\,dt\\
&\quad= C\int_Q \Big(zp\cdot(\xi_1u)_t+(a-b)zp\cdot\xi_1u+4zp_{xxx}\cdot(\xi_1u)_x+6zp_{xx}\cdot(\xi_1u)_{xx}\\
&\qquad+4zp_x\cdot(\xi_1u)_{xxx}+zp\cdot(\xi_1u)_{xxxx}\Big)\,dx\,dt\\
&\quad=: C(I_1+I_2+I_3+I_4+I_5+I_6).
\end{aligned}$$

Since

$$\begin{aligned}
|\xi_1u| &= \lambda^7e^{2\lambda a}t^{-7}(T-t)^{-7}\xi_1\\
|(\xi_1u)_t| &\le C\lambda^8e^{2\lambda a}t^{-9}(T-t)^{-9}\xi_1\\
|(\xi_1u)_x| &\le C\lambda^7e^{2\lambda a}t^{-7}(T-t)^{-7}|\xi_{1x}|+C\lambda^8e^{2\lambda a}t^{-8}(T-t)^{-8}\xi_1\\
|(\xi_1u)_{xx}| &\le C\lambda^7e^{2\lambda a}t^{-7}(T-t)^{-7}|\xi_{1xx}|+C\lambda^8e^{2\lambda a}t^{-8}(T-t)^{-8}|\xi_{1x}|\\
&\quad+C\lambda^8e^{2\lambda a}t^{-8}(T-t)^{-8}\xi_1+C\lambda^9e^{2\lambda a}t^{-9}(T-t)^{-9}\xi_1\\
|(\xi_1u)_{xxx}| &\le C\lambda^7e^{2\lambda a}t^{-7}(T-t)^{-7}(|\xi_{1xxx}|+\lambda t^{-1}(T-t)^{-1}|\xi_{1xx}|\\
&\quad+\lambda^2t^{-2}(T-t)^{-2}|\xi_{1x}|+\lambda^2t^{-2}(T-t)^{-2}|\xi_1|+\lambda t^{-1}(T-t)^{-1}|\xi_{1x}|\\
&\quad+\lambda^3t^{-3}(T-t)^{-3}|\xi_1|+\lambda t^{-1}(T-t)^{-1}|\xi_1|)\\
|(\xi_1u)_{xxxx}| &\le Ce^{2\lambda a}(\lambda^9t^{-9}(T-t)^{-9}|\xi_{1xx}|+\lambda^{10}t^{-10}(T-t)^{-10}|\xi_{1x}|\\
&\quad+\lambda^{11}t^{-11}(T-t)^{-11}|\xi_1|+\lambda^8t^{-8}(T-t)^{-8}|\xi_{1xxx}|+|\xi_{1xxxx}|),
\end{aligned}$$

by the Cauchy–Schwartz inequality and following the ideas in [5], it holds that for sufficiently large λ_0

$$\begin{aligned}
|I_1| &= \left|\int_Q zp\cdot(\xi_1u)_t\,dx\,dt\right|\\
&\le \delta\int_Q \lambda^7\theta^2t^{-7}(T-t)^{-7}p^2\xi_1\,dx\,dt+C(\delta)\int_Q \lambda^9\theta^2t^{-11}(T-t)^{-11}z^2\xi_1\,dx\,dt,\\
|I_2| &= \left|\int_Q (a-b)zp\cdot\xi_1u\,dx\,dt\right|\\
&\le \delta\int_Q \lambda^7\theta^2t^{-7}(T-t)^{-7}p^2\xi_1\,dx\,dt+C(\delta)\int_Q \lambda^7\theta^2t^{-7}(T-t)^{-7}z^2\xi_1\,dx\,dt,\\
|I_3| &= \left|\int_Q zp_{xxx}\cdot(\xi_1u)_x\,dx\,dt\right|\\
&\le \delta\int_Q \lambda\theta^2t^{-1}(T-t)^{-1}p_{xxx}^2\xi_1\,dx\,dt+C(\delta)\int_Q \lambda^{15}\theta^2t^{-15}(T-t)^{-15}z^2\chi_{B_2}\,dx\,dt,
\end{aligned}$$

$$
\begin{aligned}
|I_4| &= \left| \int_Q z p_{xx} \cdot (\xi_1 u)_{xx} \, dx \, dt \right| \\
&\le \delta \int_Q \lambda^3 \theta^2 t^{-3} (T-t)^{-3} p_{xx}^2 \xi_1 \, dx \, dt + C(\delta) \int_Q \lambda^{15} \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt, \\
|I_5| &= \left| \int_Q z p_x \cdot (\xi_1 u)_{xxx} \, dx \, dt \right| \\
&\le \delta \int_Q \lambda^5 \theta^2 t^{-5} (T-t)^{-5} p_x^2 \xi_1 \, dx \, dt + C(\delta) \int_Q \lambda^{15} \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt, \\
|I_6| &= \left| \int_Q z p \cdot (\xi_1 u)_{xxxx} \, dx \, dt \right| \\
&\le \delta \int_Q \lambda^7 \theta^2 t^{-7} (T-t)^{-7} p^2 \xi_1 \, dx \, dt + C(\delta) \int_Q \lambda^{13} \theta^2 t^{-13} (T-t)^{-13} z^2 \chi_{B_2} \, dx \, dt
\end{aligned}
$$

with $\lambda \ge \lambda_0$. Thus

$$
\begin{aligned}
&\int_Q \lambda^7 \theta^2 t^{-7} (T-t)^{-7} p^2 \, dx \, dt + \int_Q \lambda^5 \theta^2 t^{-5} (T-t)^{-5} p_x^2 \, dx \, dt \\
&\qquad + \int_Q \lambda^3 \theta^2 t^{-3} (T-t)^{-3} p_{xx}^2 \, dx \, dt + \int_Q \lambda^1 \theta^2 t^{-1} (T-t)^{-1} p_{xxx}^2 \, dx \, dt \\
&\quad \le C(I_1 + I_2 + I_3 + I_4 + I_5 + I_6) \\
&\quad \le \delta \Big(\int_Q \lambda^7 \theta^2 t^{-7} (T-t)^{-7} p^2 \xi_1 \, dx \, dt + \lambda^5 \theta^2 t^{-5} (T-t)^{-5} p_x^2 \xi_1 \, dx \, dt \\
&\qquad + \int_Q \lambda^3 \theta^2 t^{-3} (T-t)^{-3} p_{xx}^2 \xi_1 \, dx \, dt + \int_Q \lambda \theta^2 t^{-1} (T-t)^{-1} p_{xxx}^2 \xi_1 \, dx \, dt \Big) \\
&\qquad + C(\delta) \int_Q \lambda^{15} \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\int_Q \lambda^7 \theta^2 t^{-7} (T-t)^{-7} p^2 \, dx \, dt + \int_Q \lambda^5 \theta^2 t^{-5} (T-t)^{-5} p_x^2 \, dx \, dt \\
&\qquad + \int_Q \lambda^3 \theta^2 t^{-3} (T-t)^{-3} p_{xx}^2 \, dx \, dt + \int_Q \lambda \theta^2 t^{-1} (T-t)^{-1} p_{xxx}^2 \, dx \, dt \\
&\quad \le C \int_Q \lambda^{15} \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt,
\end{aligned}
$$

namely,

$$
\int_Q \theta^2 t^{-7} (T-t)^{-7} p^2 \, dx \, dt \le C \int_Q \lambda^8 \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt. \tag{3.5}
$$

From (3.4) and (3.5), we can deduce that for sufficient large λ_0

$$
\begin{aligned}
&\int_Q \lambda^7 \theta^2 t^{-7} (T-t)^{-7} z^2 \, dx \, dt \\
&\quad \le C \left(\int_{B_2 \times (0,T)} \lambda^7 \theta^2 t^{-7} (T-t)^{-7} z^2 \, dx \, dt + \int_Q \theta^2 p \chi_{\mathcal{O}}^2 \, dx \, dt \right) \\
&\quad \le C \left(\int_{B_2 \times (0,T)} \lambda^7 \theta^2 t^{-7} (T-t)^{-7} z^2 \, dx \, dt + \lambda^7 \int_Q \theta^2 t^{-7} (T-t)^{-7} p \chi_{\mathcal{O}}^2 \, dx \, dt \right) \\
&\quad \le C \left(\int_{B_2 \times (0,T)} \lambda^7 \theta^2 t^{-7} (T-t)^{-7} z^2 \, dx \, dt + \int_Q \lambda^{15} \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt \right) \\
&\quad \le C(T) \int_Q \lambda^{15} \theta^2 t^{-15} (T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt
\end{aligned}
$$

with $\lambda \geq \lambda_0$. Namely

$$\int_Q \theta^2 t^{-7}(T-t)^{-7} z^2 \, dx \, dt \leq C(T) \int_Q \lambda^8 \theta^2 t^{-15}(T-t)^{-15} z^2 \chi_{B_2} \, dx \, dt. \tag{3.6}$$

Set $\lambda \geq \lambda_1 := \max\left\{\frac{7T^2}{2M_0}, \frac{15T^2}{8m_0}, \lambda_0\right\}$ and $M_\lambda = \frac{2\lambda M_0}{T}$. On the one hand, according to Lemma 3.2, (3.6) and the definition of θ, we have

$$\begin{aligned}
&\int_{I\times(0,\frac{T}{2})} e^{-\frac{M_\lambda}{t}} z^2 \, dx \, dt \\
&\quad \leq C(T) \int_{I\times(0,\frac{T}{2})} \theta^2 t^{-7}(T-t)^{-7} z^2 \, dx \, dt \\
&\quad \leq C(T) \int_Q \theta^2 t^{-7}(T-t)^{-7} z^2 \, dx \, dt \\
&\quad \leq C(T) \int_{B_2\times(0,T)} \lambda^8 \theta^2 t^{-15}(T-t)^{-15} z^2 \, dx \, dt.
\end{aligned}$$

On the other hand, from (3.1), (3.2) and (3.5), it holds that

$$\begin{aligned}
&\int_{I\times(\frac{T}{2},T)} e^{-\frac{M_\lambda}{t}} z^2 \, dx \, dt \\
&\quad \leq \int_{I\times(\frac{T}{2},T)} z^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+1)} \int_{\mathcal{O}\times(\frac{T}{2},T)} p^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+1)} \int_{I\times(\frac{T}{2},T)} p^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)} \int_{I\times(\frac{T}{4},\frac{3T}{4})} p^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)} \int_{I\times(\frac{T}{4},\frac{3T}{4})} \theta^2 t^{-7}(T-t)^{-7} p^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)} \int_{I\times(0,T)} \theta^2 t^{-7}(T-t)^{-7} p^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)} \int_{B_2\times(0,T)} \lambda^8 \theta^2 t^{-15}(T-t)^{-15} z^2 \, dx \, dt.
\end{aligned}$$

Thus, in view of Lemma 3.2, we have

$$\begin{aligned}
&\int_Q e^{-\frac{M_\lambda}{t}} z^2 \, dx \, dt \\
&\quad = \int_{I\times(0,\frac{T}{2})} e^{-\frac{M_\lambda}{t}} z^2 \, dx \, dt + \int_{I\times(\frac{T}{2},T)} e^{-\frac{M_\lambda}{t}} z^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)} \int_{B_2\times(0,T)} \lambda^8 \theta^2 t^{-15}(T-t)^{-15} z^2 \, dx \, dt \\
&\quad \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)} \int_{B_2\times(0,T)} z^2 \, dx \, dt.
\end{aligned} \tag{3.7}$$

Finally, setting $\lambda = \lambda_1$ in (3.7) and we define $M = M_{\lambda_1}$.

By a density argument, (3.7) holds for the solution (p, z) of (1.7)–(1.8) if the initial data $p^0 \in L^2(I)$. Indeed, we can choose a sequence $\{p_n^0\} \subset H_0^2(I)$ such that $p_n^0 \to p^0$ in $L^2(I)$. By i) of Proposotion 2.3, we obtain that $\int_Q e^{-\frac{M}{t}} z_n^2 \, dx \, dt \to \int_Q e^{-\frac{M}{t}} z^2 \, dx \, dt$ and $\int_{\omega\times(0,T)} z_n^2 \, dx \, dt \to$

$\int_{\omega\times(0,T)} z^2\,dx\,dt$, where z_n and z are the solutions of (1.7)–(1.8) with the initial data p_n^0 and p^0, respectively. Since $\int_Q e^{-\frac{M}{t}} z_n^2\,dx\,dt \le C\int_{\omega\times(0,T)} z_n^2\,dx\,dt$, by passing to the limit $n\to\infty$, (3.7) holds for the initial data $p^0 \in L^2(I)$.

4 Proof of Theorem 1.3

In this section, we set $y^0 = 0$. In order to establish the null controllability property of (1.3)–(1.4), we firstly consider the null controllability property of (1.5)–(1.6).

We define the following functional:

$$J_\varepsilon\colon L^2(I)\to\mathbb{R}$$
$$J_\varepsilon(p^0) = \frac{1}{2}\int_0^T\int_\omega z^2\,dx\,dt + \varepsilon\|p^0\|_{L^2(I)} + \int_0^T\int_I \xi z\,dx\,dt$$

where z is the solution of the adjoint system (1.7)–(1.8).

The following proposition ensures that the minimum of J_ε gives a control for the null controllability property of (1.5)–(1.6).

Proposition 4.1. *Given $\varepsilon > 0$. If $\hat{p}_\varepsilon^0$ is a minimum point of J_ε in $L^2(I)$ and $\hat{z}_\varepsilon$ is the solution of* (1.7)–(1.8) *with initial data $\hat{p}_\varepsilon^0$, then $h = \hat{z}_\varepsilon$ is a control for* (1.5)–(1.6) *such that*

$$\|q_\varepsilon(x,0)\|_{L^2(I)} < \varepsilon. \tag{4.1}$$

Proof. For reasons of simplicity, we denote $\hat{p}_\varepsilon^0$, $\hat{z}_\varepsilon$, q_ε by $\hat{p}^0$, $\hat{z}$, q.

For any $p^1 \in L^2(I)$ and $s \in R$, the following inequality holds

$$\begin{aligned}
0 &\le J_\varepsilon(sp^1+\hat{p}^0) - J_\varepsilon(\hat{p}^0)\\
&= \frac{1}{2}\int_0^T\int_I \left((sz^1+\hat{z}^0)^2 - (\hat{z}^0)^2\right)\,dx\,dt + s\int_0^T\int_I \xi z^1\,dx\,dt\\
&\quad + \varepsilon\left(\|sp^1+\hat{p}^0\|_{L^2(I)} - \|\hat{p}^0\|_{L^2(I)}\right)\\
&= \frac{1}{2}\int_0^T\int_I \left(s^2(z^1)^2 + 2sz^1\hat{z}^0\right)\,dx\,dt + s\int_0^T\int_I \xi z^1\,dx\,dt\\
&\quad + \varepsilon\left(\|sp^1+\hat{p}^0\|_{L^2(I)} - \|\hat{p}^0\|_{L^2(I)}\right),
\end{aligned}$$

where z^1 is the solution of (1.7)–(1.8) with initial data p^1.

Since $\|sp^1+\hat{p}^0\|_{L^2(I)} - \|\hat{p}^0\|_{L^2(I)} \le |s|\|p^1\|_{L^2(I)}$, we obtain

$$0 \le \frac{1}{2}\int_0^T\int_I \left(s^2(z^1)^2 + 2sz^1\hat{z}^0\right)\,dx\,dt + s\int_0^T\int_I \xi z^1\,dx\,dt + \varepsilon|s|\|p^1\|_{L^2(I)}.$$

Dividing by $s > 0$ and by passing to the limit $s \to 0$, it holds that

$$0 \le \int_0^T\int_I z^1(\hat{z}^0+\xi)\,dx\,dt + \varepsilon\|p^1\|_{L^2(I)}.$$

The same calculations with $s < 0$ gives that

$$0 \le \int_0^T\int_I -z^1(\hat{z}^0+\xi)\,dx\,dt + \varepsilon\|p^1\|_{L^2(I)}.$$

Namely,

$$\left| \int_0^T \int_I z^1(\hat{z}^0 + \xi)\, dx\, dt \right| \le \varepsilon \|p^1\|_{L^2(I)}.$$

If we take $h = \hat{z}$ in (1.3)–(1.4), by (1.10), we know

$$\int_Q z^1(\hat{z}|_\omega + \xi)\, dx\, dt = \int_I q(x,0) p^1(x)\, dx.$$

Thus

$$\left| \int_I q(x,0) p^1(x)\, dx \right| \le \varepsilon \|p^1\|_{L^2(I)}, \quad \forall\, p^1 \in L^2(I),$$

namely

$$\|q(x,0)\|_{L^2(I)} < \varepsilon.$$

□

Proposition 4.2. *For any $\varepsilon > 0$, if $\hat{p}_\varepsilon^0$ is a minimum point of J_ε in $L^2(I)$ and $\hat{z}_\varepsilon$ is the solution of (1.7)-(1.8) with initial data $\hat{p}_\varepsilon^0$, then*

$$\|\hat{z}_\varepsilon\|_{L^2(\omega\times(0,T))} \le e^{C(T)(\|a\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)} + 1)} \|e^{\frac{M}{2t}} \xi\|_{L^2(Q)}. \tag{4.2}$$

Proof. It is easy to see that

$$\begin{aligned}
0 = J_\varepsilon(0) &\ge J_\varepsilon(\hat{p}_\varepsilon^0) \\
&= \frac{1}{2} \int_0^T \int_\omega (\hat{z}_\varepsilon)^2\, dx\, dt + \int_0^T \int_I \xi \hat{z}_\varepsilon\, dx\, dt + \varepsilon \|\hat{p}_\varepsilon^0\|_{L^2(I)} \\
&\ge \frac{1}{2} \int_0^T \int_\omega (\hat{z}_\varepsilon)^2\, dx\, dt - \|e^{\frac{M}{2t}} \xi\|_{L^2(Q)} \|e^{\frac{-M}{2t}} \hat{z}_\varepsilon\|_{L^2(Q)} + \varepsilon \|\hat{p}_\varepsilon^0\|_{L^2(I)} \\
&\ge \frac{1}{2} \int_0^T \int_\omega (\hat{z}_\varepsilon)^2\, dx\, dt - e^{C(T)(\|a\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)} + 1)} \|e^{\frac{M}{2t}} \xi\|_{L^2(Q)} \|\hat{z}_\varepsilon\|_{L^2(\omega\times(0,T))}.
\end{aligned}$$

Then, we have

$$\|\hat{z}_\varepsilon\|_{L^2(\omega\times(0,T))} \le e^{C(T)(\|a\|_{L^\infty(Q)} + \|b\|_{L^\infty(Q)} + 1)} \|e^{\frac{M}{2t}} \xi\|_{L^2(Q)}.$$

□

Proposition 4.3. *$J_\varepsilon(\cdot)$ is continuous, strictly convex and*

$$\liminf_{\|p^0\|_{L^2(I)} \to \infty} \frac{J_\varepsilon(p^0)}{\|p^0\|_{L^2(I)}} \ge \varepsilon.$$

Proof. The continuity and strict convexity can be obtained easily. Next, we show that $J_\varepsilon(\cdot)$ is coercive.

Let $p_n^0 \in L^2(I), \|p_n^0\|_{L^2(I)} \to \infty$.

Define

$$v_n^0 = \frac{p_n^0}{\|p_n^0\|_{L^2(I)}}$$

and (v_n, w_n) denotes the corresponding solution to (1.7)–(1.8) with $p^0 = v_n^0$. Then $\|v_n^0\|_{L^2(I)} = 1$, by (i) of Proposition 2.3,

$$\|v_n\|_{C([0,T];L^2(I))\cap L^2(0,T;H_0^2(I)))} \le C, \qquad \|w_n\|_{C([0,T];L^2(I))\cap L^2(0,T;H_0^2(I)))} \le C,$$

where C is a constant which is independent of n.

According to $v_{nt} = -v_{nxxxx} - bv$ and $w_{nt} = -w_{nxxxx} - aw + v\chi_{\mathcal{O}}$, $\{v_{nt}\}$ and $\{w_{nt}\}$ are bounded in $L^2(0,T;H^{-2}(I))$. Applying Aubin's compactness theorem (see, for instance, [25]), we obtain a subsequence (still denoted by n) such that

$$\begin{aligned} v_n^0 \rightharpoonup v^0 \quad & \text{weakly in} \quad L^2(I), \\ v_n \rightarrow v \quad & \text{strongly in} \quad L^2(Q), \\ w_n \rightarrow w \quad & \text{strongly in} \quad L^2(Q). \end{aligned}$$

On the other hand,

$$\frac{J_\varepsilon(p_n^0)}{\|p_n^0\|_{L^2(I)}} = \frac{\|p_n^0\|_{L^2(I)}}{2}\int_0^T\int_I w_n^2\,dx\,dt + \int_0^T\int_I \xi w_n\,dx\,dt + \varepsilon.$$

The following two cases may occur:

(i) $\liminf\limits_{n\to\infty} \int_0^T\int_I w_n^2\,dx\,dt > 0$. In this case we obtain immediately that

$$\liminf_{\|p^0\|_{L^2(I)}\to\infty} \frac{J_\varepsilon(p^0)}{\|p^0\|_{L^2(I)}} \geq \varepsilon.$$

(ii) $\liminf\limits_{n\to\infty} \int_0^T\int_I w_n^2\,dx\,dt = 0$. In this case, since $v_n^0 \rightharpoonup v^0$ weakly in $L^2(I)$, $v_n \rightarrow v$ strongly in $L^2(Q)$ and $w_n \rightarrow w$ strongly in $L^2(Q)$, (v,w) is the solution of (1.7)–(1.8) with initial data $p^0 = v^0$. Moreover, by the lower semi-continuity

$$\int_0^T\int_I w^2\,dx\,dt \leq \liminf_{n\to\infty}\int_0^T\int_I w_n^2\,dx\,dt = 0.$$

Therefore, $\omega = 0$, namely $w_n \rightarrow 0$ strongly in $L^2(Q)$. Consequently $\int_0^T\int_I \xi w_n\,dx\,dt \rightarrow 0$. Hence,

$$\liminf_{n\to\infty}\frac{J_\varepsilon(p_n^0)}{\|p_n^0\|_{L^2(I)}} \geq \liminf_{n\to\infty}\left(\int_0^T\int_I \xi w_n\,dx\,dt + \varepsilon\right) = \varepsilon.$$

□

Then, we can obtain the following result:

Proposition 4.4. *System* (1.5)–(1.6) *with initial data* $y^0 = 0$ *is null controllable. Moreover, the control* h *satisfies*

$$\|h\|_{L^2(\omega\times(0,T))} \leq e^{C(T)(\|a\|_{L^\infty(Q)}+\|b\|_{L^\infty(Q)}+1)}\|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}. \tag{4.3}$$

Proof. For any $\varepsilon > 0$, there exist a control $\hat{z}_\varepsilon \in L^2(Q)$ satisfying (4.2) and q_ε satisfying (4.1), where $(y_\varepsilon, q_\varepsilon)$ is the solution of (1.5)–(1.6) with initial data $y^0 = 0$ and $h = z_\varepsilon$.

From (4.2), by extracting subsequences, still denoted in the same way, we have that there exists a function $z \in L^2(\omega\times(0,T))$ such that $\hat{z}_\varepsilon \rightharpoonup z$ in $L^2(\omega\times(0,T))$. Let $h = z$. Combining (4.1), (4.2) and (4.3), the solution q to (1.5)–(1.6) with $h = z$ as the control satisfies $q(\cdot,T) = 0$, and h satisfies (4.3). □

We now apply a fixed point argument to prove a insensitivity result in the nonlinear case.
Proof of Theorem 1.3

We may as well assume that f is in $C^1(R)$ and we shall use a fixed point argument applying Kakutani's theorem. The general case of a function f can be easily obtained by a density argument (see [3, 17]).

Let

$$g(s) = \begin{cases} \dfrac{f(s)}{s}, & s \neq 0, \\ f'(0), & s = 0. \end{cases}$$

Then g is continuous in R and for each $\varepsilon > 0$, there exists a positive constant C_ε (which only depends on ε and on the function f) such that

$$|g(s)| + |f'(s)| \leq C_\varepsilon + \varepsilon \log(1 + |s|) \tag{4.4}$$

for all $s \in R$.

Set $X = L^\infty(Q)$.

For any $\eta \in \overline{B}(0, R) \subset X$, $R > 0$ to be determined later, we consider the following system

$$\begin{cases} y_t + y_{xxxx} + g(\eta)y = \xi + h|_\omega & \text{in } Q, \\ y(0,t) = 0 = y(1,t) & \text{in } (0,T), \\ y_x(0,t) = 0 = y_x(1,t) & \text{in } (0,T), \\ y(x,0) = 0 & \text{in } I \end{cases} \tag{4.5}$$

$$\begin{cases} -q_t + q_{xxxx} + f'(\eta)q = y|_{\mathcal{O}} & \text{in } Q, \\ q(0,t) = 0 = q(1,t) & \text{in } (0,T), \\ q_x(0,t) = 0 = q_x(1,t) & \text{in } (0,T), \\ q(x,T) = 0 & \text{in } I. \end{cases} \tag{4.6}$$

In accordance with the results in Proposition 4.4, for any $\eta \in X$, there exists $h_\eta \in L^2(\omega \times (0,T))$ such that $q_\eta(\cdot, 0) = 0$, and h_η satisfies

$$\|h_\eta\|_{L^2(\omega\times(0,T))} \leq e^{C(T)(\|g(\eta)\|_{L^\infty(Q)} + \|f'(\eta)\|_{L^\infty(Q)} + 1)} \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}.$$

It follows from Proposition 2.3 that

$$\begin{aligned} &\|y\|_{C([0,T];H_0^2(I))\cap L^2(0,T;H^4(I))} \\ &\quad\leq C(T)e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+1)}\|\xi + h|_\omega\|_{L^2(Q)} \\ &\quad\leq C(T)e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+1)}(\|\xi\|_{L^2(Q)} + \|h|_\omega\|_{L^2(Q)}) \\ &\quad\leq C(T)e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+1)}\left(\|\xi\|_{L^2(Q)} + e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+\|f'(\eta)\|_{L^\infty(Q)}+1)}\|e^{\frac{M}{2t}}\xi|_{L^2(Q)}\right) \\ &\quad\leq C(T)e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+\|f'(\eta)\|_{L^\infty(Q)}+1)}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}). \end{aligned} \tag{4.7}$$

Now, for each $\eta \in \overline{B}(0, R)$, set

$$\begin{aligned} H(\eta) =& \Big\{h_\eta \in L^2(\omega \times (0,T)) : (y,q) \text{ is the solution of (4.5)–(4.6)}, \ q_\eta(\cdot,0) = 0, \\ &\qquad \text{and } \|h_\eta\|_{L^2(\omega\times(0,T))} \leq e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+\|f'(\eta)\|_{L^\infty(Q)}+1)}\|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}\Big\}, \\ \Lambda(\eta) =& \{y : (y,q) \text{ is the solution of (4.5)–(4.6) with } h_\eta \in H(\eta)\}. \end{aligned}$$

In this way, we have been able to introduce a set-valued mapping on X :

$$\Lambda \colon \eta \in X \to \Lambda(\eta) \subset X.$$

We shall prove that this mapping possesses at least one fixed point y.

Let us prove that Λ fulfills the assumptions of Kakutani's fixed-point theorem.

In the first place, one can check that $\Lambda(\eta)$ is a nonempty closed convex subset of X for fixed $z \in X_0$, due to the linearity of system (4.5)-(4.6).

According to estimate (4.7), $\Lambda(\eta)$ is a bounded set in $C([0,T];H_0^2(I)) \cap L^2(0,T;H^4(I))$. Since $C([0,T];H_0^2(I)) \cap L^2(0,T;H^4(I)) \subset L^2(0,T;H^4(I)) \cap H^1(0,T;L^2(I)) \hookrightarrow C^{\beta,\frac{\beta}{2}}$ with $\beta = \frac{1}{2}$, it follows that each $\Lambda(\eta)$ is a compact subset in X.

In the second place, Λ is upper semicontinuous. Indeed, if $\{\eta_n\} \subset X$, $y_n \in \Lambda(\eta_n)$, $\eta_n \to \eta$ in X, and $y_n \to y$ in X, by using the regularity of the solution of (4.5)–(4.6), extracting subsequences, still denoted in the same way, there exist $q \in C([0,T], L^2(I)) \cap L^2(0,T;H_0^2(I))$ and $h \in L^2(\omega \times (0,T))$, such that $q_n \to q$ in $C([0,T], L^2(I)) \cap L^2(0,T;H_0^2(I))$ and $h_n \rightharpoonup h$ in $L^2(\omega \times (0,T))$, then (y,q,h) satisfies (4.5)–(4.6) corresponding to $h \in L^2(\omega \times (0,T))$, and $q(\cdot,0) = 0$ in Ω, namely $y \in \Lambda(\eta)$.

Finally, let us see that there exists $R > 0$ such that $\Lambda(\overline{B}(0,R)) \subset \overline{B}(0,R)$. Indeed, for any $\eta \in \overline{B}(0,R)$, from (4.7) and (4.4) it is observed that each $y \in \Lambda(\eta)$ satisfies

$$\begin{aligned}
&\|y\|_{C([0,T];H_0^2(I))\cap L^2(0,T;H^4(I))} \\
&\quad \le C(T)e^{C(T)(\|g(\eta)\|_{L^\infty(Q)}+\|f'(\eta)\|_{L^\infty(Q)}+1)}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}) \\
&\quad \le C(T)e^{C(T)(C_\varepsilon+\varepsilon\log(1+\|\eta\|_{L^\infty(Q)})+1)}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}) \\
&\quad \le e^{C(T)(C_\varepsilon+\varepsilon\log(1+\|\eta\|_{L^\infty(Q)})+1)}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi|_{L^2(Q)}) \\
&\quad = e^{C(T)(C_\varepsilon+1)}(1+R)^{\varepsilon C(T)}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}).
\end{aligned}$$

Thus, choosing $\varepsilon = \frac{1}{2C(T)}$, we obtain

$$\|y\|_{C([0,T];H_0^2(I))\cap L^2(0,T;H^4(I))} \le C(1+R)^{\frac{1}{2}}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}),$$

from which we infer the existence of $R > 0$ large enough such that

$$\begin{aligned}
\|y\|_X &\le C\|y\|_{C([0,T];H_0^2(I))\cap L^2(0,T;H^4(I))} \\
&\le C(1+R)^{\frac{1}{2}}(\|\xi\|_{L^2(Q)} + \|e^{\frac{M}{2t}}\xi\|_{L^2(Q)}) \\
&\le R.
\end{aligned}$$

Namely, $\Lambda(\overline{B}(0,R)) \subset \overline{B}(0,R)$.

By the Kakutani's fixed point theorem (see, for instance, [1]), Theorem 1.3 follows.

5 Appendix A: Proof of Proposition 2.1

Set $u = \theta y, Py = f$. Direct computation shows that

$$\theta(y_t + y_{xxxx}) = u_t + A_0 u + A_1 u_x + A_2 u_{xx} + A_3 u_{xxx} + u_{xxxx}, \tag{5.1}$$

where

$$\begin{aligned}
A_0 &= l_x^4 + 4l_x l_{xxx} - l_{xxxx} - 6l_x^2 l_{xx} + 3l_{xx}^2 - l_t,\\
A_1 &= -4l_x^3 + 12 l_x l_{xx} - 4l_{xxx},\\
A_2 &= 6l_x^2 - 6l_{xx},\\
A_3 &= -4l_x.
\end{aligned}$$

Set

$$\begin{aligned}
I_1 &= u_t + B_1 u_x + B_3 u_{xxx} + Eu,\\
I_2 &= u_{xxxx} + B_0 u + B_2 u_{xx} + F u_x,\\
R &= \theta f - I_1 - I_2 = S_0 u + S_1 u_x + S_2 u_{xx},
\end{aligned}$$

where

$$\begin{aligned}
&B_0 = l_x^4, \quad B_1 = -4l_x^3, \quad B_2 = 6l_x^2, \quad B_3 = -4l_x,\\
&E = -4l_x^2 l_{xx}, \quad F = 12 l_x l_{xx}, \quad S_0 = 4l_x l_{xxx} - l_{xxxx} - 6l_x^2 l_{xx} + 3l_{xx}^2 - l_t,\\
&S_1 = 12 l_x l_{xx} - 4l_{xxx}, \quad S_2 = -6l_{xx}.
\end{aligned}$$

Step 1. We shall prove the following equality

$$\begin{aligned}
I_1 \cdot I_2 = u^2\{\cdots\} + u_x^2\{\cdots\} + u_{xx}^2\{\cdots\} + u_{xxx}^2\{\cdots\}\\
+ \{\cdots\}_x + \{\cdots\}_{xx} + \{\cdots\}_{xxx} + \{\cdots\}_t,
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
\{\cdots\}_x &= \Big\{u_t u_{xxx} - u_{xt} u_{xx} + B_2 u_t u_x + \frac{3}{2} B_{1xx} u_x^2 - \frac{3}{2} B_1 u_{xx}^2 + \frac{1}{2} B_1 B_2 u_x^2\\
&\quad + \frac{1}{2} B_0 B_1 u^2 + \frac{1}{2} B_3 u_{xxx}^2 + \frac{1}{2} B_2 B_3 u_{xx}^2 + \frac{3}{2}(B_0 B_3)_{xx} u^2 - \frac{3}{2} B_0 B_3 u_x^2\\
&\quad + 4E_x u_x^2 - 2E_{xxx} u^2 - (B_2 E)_x u^2 + \frac{1}{2} E F u^2 - (B_3 F)_x u_x^2\Big\}_x,\\
\{\cdots\}_{xx} &= \Big\{-\frac{3}{2} B_{1x} u_x^2 - \frac{3}{2}(B_0 B_3)_x u^2 + 3E_{xx} u^2 - 2E u_x^2 + \frac{1}{2} B_2 E u^2 + \frac{1}{2} B_3 F u_x^2\Big\}_{xx},\\
\{\cdots\}_{xxx} &= \Big\{\frac{1}{2} B_1 u_x^2 + \frac{1}{2} B_0 B_3 u^2 - 2E_x u^2\Big\}_{xxx},\\
\{\cdots\}_{xxxx} &= \Big\{\frac{1}{2} E u^2\Big\}_{xxxx},\\
\{\cdots\}_t &= \Big\{\frac{1}{2} B_0 u^2 - \frac{1}{2} B_2 u_x^2 + \frac{1}{2} u_{xx}^2\Big\}_t,\\
u^2\{\cdots\} &= u^2\Big\{-\frac{1}{2} B_{0t} - \frac{1}{2}(B_0 B_1)_x - \frac{1}{2}(B_0 B_3)_{xxx} + \frac{1}{2} E_{xxxx} + \frac{1}{2}(B_2 E)_{xx}\\
&\quad + B_0 E - \frac{1}{2}(EF)_x\Big\},\\
u_x^2\{\cdots\} &= u_x^2\Big\{\frac{1}{2} B_{2t} - \frac{1}{2} B_{1xxx} - \frac{1}{2}(B_1 B_2)_x + \frac{3}{2}(B_0 B_3)_x - 2E_{xx} - B_2 E\\
&\quad + B_1 F + \frac{1}{2}(B_3 F)_{xx}\Big\},\\
u_{xx}^2\{\cdots\} &= u_{xx}^2\Big\{\frac{3}{2} B_{1x} - \frac{1}{2}(B_2 B_3)_x + E - B_3 F\Big\},\\
u_{xxx}^2\{\cdots\} &= u_{xxx}^2\Big\{-\frac{1}{2} B_{3x}\Big\}.
\end{aligned}$$

Indeed, (5.2) can be obtained from the following equations.

$$
\begin{aligned}
u_t \cdot u_{xxxx} &= (u_t u_{xxx} - u_{xt} u_{xx})_x + \frac{1}{2}(u_{xx}^2)_t,\\
u_t \cdot B_2 u_{xx} &= -B_{2x} u_t u_x + \frac{1}{2} B_{2t} u_x^2 - \frac{1}{2}(B_2 u_x^2)_t + (B_2 u_t u_x)_x,\\
u_t \cdot B_0 u &= \frac{1}{2}\Big((B_0 u^2)_t - B_{0t} u^2\Big),\\
u_t \cdot F u_x &= F u_t u_x,\\
B_1 u_x \cdot u_{xxxx} &= \frac{1}{2}(B_1 u_x^2)_{xxx} - \frac{3}{2}(B_{1x} u_x^2)_{xx} + \Big(\frac{3}{2} B_{1xx} u_x^2 - \frac{3}{2} B_1 u_{xx}^2\Big)_x\\
&\quad + \frac{3}{2} B_{1x} u_{xx}^2 - \frac{1}{2} B_{1xxx} u_x^2,\\
B_1 u_x \cdot B_2 u_{xx} &= B_1 B_2 u_x u_{xx} = \frac{1}{2}\Big((B_1 B_2 u_x^2)_x - (B_1 B_2)_x u_x^2\Big),\\
B_1 u_x \cdot B_0 u &= B_0 B_1 u u_x = \frac{1}{2}\Big((B_0 B_1 u^2)_x - (B_0 B_1)_x u^2\Big),\\
B_1 u_x \cdot F u_x &= B_1 F u_x^2,\\
B_3 u_{xxx} \cdot u_{xxxx} &= \frac{1}{2}\Big((B_3 u_{xxx}^2)_x - B_{3x} u_{xxx}^2\Big),\\
B_3 u_{xxx} \cdot B_2 u_{xx} &= B_2 B_3 u_{xx} u_{xxx} = \frac{1}{2}\Big((B_2 B_3 u_{xx}^2)_x - (B_2 B_3)_x u_{xx}^2\Big),\\
B_3 u_{xxx} \cdot B_0 u &= B_0 B_3 u u_{xxx} = \frac{1}{2}(B_0 B_3 u^2)_{xxx} - \frac{3}{2}\left((B_0 B_3)_x u^2\right)_{xx}\\
&\quad + \Big(\frac{3}{2}(B_0 B_3)_{xx} u^2 - \frac{3}{2} B_0 B_3 u_x^2\Big)_x + \frac{3}{2}(B_0 B_3)_x u_x^2 - \frac{1}{2}(B_0 B_3)_{xxx} u^2,\\
B_3 u_{xxx} \cdot F u_x &= B_3 F u_x u_{xxx} = \frac{1}{2}(B_3 F u_x^2)_{xx} - \big((B_3 F)_x u_x^2\big)_x - B_3 F u_{xx}^2 + \frac{1}{2}(B_3 F)_{xx} u_x^2,\\
E u \cdot u_{xxxx} &= \frac{1}{2}(E u^2)_{xxxx} - 2(E_x u^2)_{xxx} + (3E_{xx} u^2 - 2E u_x^2)_{xx}\\
&\quad + (4E_x u_x^2 - 2E_{xxx} u^2)_x + E u_{xx}^2 - 2E_{xx} u_{xx}^2 + \frac{1}{2} E_{xxxx} u^2,\\
E u \cdot B_2 u_{xx} &= B_2 E u u_{xx} = \frac{1}{2}(B_2 E u^2)_{xx} - \big((B_2 E)_x u^2\big)_x - B_2 E u_x^2 + \frac{1}{2}(B_2 E)_{xx} u^2,\\
E u \cdot B_0 u &= B_0 E u^2,\\
E u \cdot F u_x &= E F u u_x = \frac{1}{2}\left((E F u^2)_x - (E F)_x u^2\right).
\end{aligned}
$$

Step 2. We shall prove the following estimate

$$
\begin{aligned}
&\int_Q \Big(\frac{1}{\lambda\varphi}\theta^2(y_t^2 + y_{xxxx}^2) + \lambda\varphi\theta^2 y_{xxx}^2 + \lambda^3\varphi^3\theta^2 y_{xx}^2 + \lambda^5\varphi^5\theta^2 y_x^2 + \lambda^7\varphi^7\theta^2 y^2\Big)\,dx\,dt\\
&\quad \le C\left(\int_{Q^{\omega_0}} (\lambda\varphi\theta^2 y_{xxx}^2 + \lambda^3\varphi^3\theta^2 y_{xx}^2 + \lambda^5\varphi^5\theta^2 y_x^2 + \lambda^7\varphi^7\theta^2 y^2)\,dx\,dt + \int_Q \theta^2 f^2\,dx\,dt\right).
\end{aligned} \tag{5.3}
$$

Indeed, by the definition of a, φ, ψ and μ, it is obvious that

$$
\begin{aligned}
&|a_x| \le C(\psi)\mu\varphi, \quad |a_{xx}| \le C(\psi)\mu^2\varphi, \quad |a_{xxx}| \le C(\psi)\mu^3\varphi,\\
&|a_{xxxx}| \le C(\psi)\mu^4\varphi, \quad |a_{xxxxx}| \le C(\psi)\mu^5\varphi, \quad |a_{xxxxxx}| \le C(\psi)\mu^6\varphi,
\end{aligned}
$$

$$|a_{xxxxxxx}| \le C(\psi)\mu^7\varphi, \quad |a_{xt}| \le C(\psi)\mu T\varphi^2, \quad |a_{xxt}| \le C(\psi)T\mu^2\varphi^2,$$
$$|a_{xxxt}| \le C(\psi)\mu^3 T\varphi^2, \quad |a_{xxxxt}| \le C(\psi)T\mu^4\varphi^2, \quad |a_t| \le CT\varphi^2,$$
$$|a_{tt}| \le CT^4\varphi^4.$$

Observe that $\varphi \le \frac{T^2}{4}\varphi^2 \le \frac{T^4}{16}\varphi^3 \le \frac{T^6}{64}\varphi^4 \le \frac{T^8}{256}\varphi^5 \le \frac{T^{10}}{1024}\varphi^6$.

For the term $u^2\{\cdots\}$ in (5.2), if we choose $\lambda \ge \mu C(\psi)(T+T^2)$ with $C(\psi)$ large enough, then it holds that

$$-\frac{1}{2}B_{0t} - \frac{1}{2}(B_0B_1)_x - \frac{1}{2}(B_0B_3)_{xxx} + \frac{1}{2}E_{xxxx} + \frac{1}{2}(B_2E)_{xx} + B_0E - \frac{1}{2}(EF)_x = 10\lambda^7\mu^8\varphi^7\psi_x^8 + R_0,$$

where

$$|R_0| \le C\lambda^7\mu^7\varphi^7.$$

Namely

$$u^2\{\cdots\} = 10\lambda^7\mu^8\varphi^7\psi_x^8u^2 + R_0u^2. \tag{5.4}$$

Using the same method, we can obtain that

$$\begin{aligned} u_x^2\{\cdots\} &= 6\lambda^5\mu^6\varphi^5\psi_x^6u_x^2 + R_1u_x^2,\\ u_{xx}^2\{\cdots\} &= 62\lambda^3\mu^4\varphi^3\psi_x^4u_{xx}^2 + R_2u_{xx}^2,\\ u_{xxx}^2\{\cdots\} &= 2\lambda\mu^2\varphi\psi_x^2u_{xxx}^2 + R_3u_{xxx}^2, \end{aligned} \tag{5.5}$$

where

$$|R_1| \le C\lambda^5\mu^5\varphi^5, \quad |R_2| \le C\lambda^3\mu^3\varphi^3 \quad \text{and} \quad |R_3| \le C\lambda\mu\varphi.$$

Now, we estimate the term $\int_Q(\{\cdots\}_x + \{\cdots\}_{xx} + \{\cdots\}_{xxx} + \{\cdots\}_t)\,dx\,dt$ in (5.2).

Indeed, noting $y(0,t) = y(1,t) = y_x(0,t) = y_x(1,t) = 0$ and $\lim_{t\to 0^+}\varphi(t,\cdot) = \lim_{t\to T^-}\varphi(t,\cdot) = +\infty$, we have

$$u(0,t) = u(1,t) = u_x(0,t) = u_x(1,t) = 0 \quad \forall\, t \in (0,T)$$

and

$$u(x,0) = u(x,T) = u_x(x,0) = u_x(x,T) = u_{xx}(x,0) = u_{xx}(x,T) = 0 \quad \forall\, x \in I.$$

Then the following holds

$$\begin{aligned} &\int_Q (\{\cdots\}_x + \{\cdots\}_{xx} + \{\cdots\}_{xxx} + \{\cdots\}_t)\,dx\,dt\\ &= \int_Q \Big(u_{xx}^2(\frac{1}{2}B_2B_3 - \frac{1}{2}B_1) + u_{xxx}^2(\frac{1}{2}B_3)\Big)_x\,dx\,dt\\ &= \int_Q (u_{xx}^2(-10l_x^3) + u_{xxx}^2(-2l_x))_x\,dx\,dt\\ &\triangleq V(1) - V(0). \end{aligned}$$

Noting $\psi_x(1) > 0$ and $\psi_x(0) < 0$, we have

$$\begin{aligned} V(1) &= \int_0^T \Big(u_{xx}^2(10\lambda^3\mu^3\varphi^3\psi_x^3) + u_{xxx}^2(2\lambda\mu\varphi\psi_x)\Big)(1,t)\,dt \ge 0,\\ -V(0) &= \int_0^T \Big(u_{xx}^2(-10\lambda^3\mu^3\varphi^3\psi_x^3) + u_{xxx}^2(-2\lambda\mu\varphi\psi_x)\Big)(0,t)\,dt \ge 0. \end{aligned}$$

Thus,

$$V(1) - V(0) \ge 0. \tag{5.6}$$

Due to (5.1), we have

$$I_1 + I_2 = \theta f - S_0 u - S_1 u_x - S_2 u_{xx},$$

where

$$|S_0|^2 \leq C\lambda^7\mu^7\varphi^7, \quad |S_1|^2 \leq C\lambda^5\mu^5\varphi^5 \quad \text{and} \quad |S_2|^2 \leq C\lambda^3\mu^3\varphi^3.$$

Then we can deduce that

$$\begin{aligned}
&\int_Q (I_1^2 + I_2^2 + 2I_1 I_2)\,dx\,dt \\
&\quad = \|I_1 + I_2\|^2_{L^2(Q)} \\
&\quad = \|\theta f - S_0 u - S_1 u_x - S_2 u_{xx}\|^2_{L^2(Q)} \\
&\quad \leq C\left(\int_Q \theta^2 f^2\,dx\,dt + \int_Q (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2)\,dx\,dt\right).
\end{aligned} \tag{5.7}$$

From (5.4)–(5.7), we can obtain that

$$\begin{aligned}
&\int_Q (I_1^2 + I_2^2 + \lambda^7\mu^8\varphi^7\psi_x^8 u^2 + \lambda^5\mu^6\varphi^5\psi_x^6 u_x^2 + \lambda^3\mu^4\varphi^3\psi_x^4 u_{xx}^2 + \lambda\mu^2\varphi\psi_x^2 u_{xxx}^2)\,dx\,dt \\
&\quad \leq C\left(\int_Q \theta^2 f^2\,dx\,dt + \int_Q (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2)\,dx\,dt\right).
\end{aligned}$$

Recall that $|\psi_x| > 0$ in $\bar{I}\setminus\omega$, it follows that

$$\begin{aligned}
&\int_{Q\setminus Q^\omega} (\lambda^7\mu^8\varphi^7 u^2 + \lambda^5\mu^6\varphi^5 u_x^2 + \lambda^3\mu^4\varphi^3 u_{xx}^2 + \lambda\mu^2\varphi u_{xxx}^2)\,dx\,dt \\
&\quad \leq C(\psi)\left(\int_Q \theta^2 f^2\,dx\,dt + \int_Q (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2)\,dx\,dt\right),
\end{aligned}$$

from which if we choose $\mu_0 = C(\psi) + 1$, then it holds that

$$\begin{aligned}
&\int_{Q\setminus Q^\omega} (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\,dx\,dt \\
&\quad \leq C_1(\psi)\left(\int_Q \theta^2 f^2\,dx\,dt + \int_{Q^\omega} (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\right)dx\,dt.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_{Q\setminus Q^\omega} (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\,dx\,dt \\
&\qquad + \int_{Q^\omega} (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\,dx\,dt \\
&\quad \leq C\left(\int_{Q^\omega} (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\,dx\,dt + \int_Q \theta^2 f^2\,dx\,dt\right).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_Q (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\,dx\,dt \\
&\quad \leq C\left(\int_Q \theta^2 f^2\,dx\,dt + \int_{Q^\omega} (\lambda^7\mu^7\varphi^7 u^2 + \lambda^5\mu^5\varphi^5 u_x^2 + \lambda^3\mu^3\varphi^3 u_{xx}^2 + \lambda\mu\varphi u_{xxx}^2)\right)dx\,dt,
\end{aligned}$$

from which it holds that

$$\int_Q (\lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda^3\varphi^3 u_{xx}^2 + \lambda\varphi u_{xxx}^2)\,dx\,dt$$
$$\leq C(\mu)\left(\int_Q \theta^2 f^2\,dx\,dt + \int_{Q^\omega} (\lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda^3\varphi^3 u_{xx}^2 + \lambda\varphi u_{xxx}^2)\right) dx\,dt.$$

According to the definition of I_1 and I_2, direct computation shows

$$\begin{aligned}\int_Q \frac{1}{\lambda\varphi} u_{xxxx}^2\,dx\,dt &= \int_Q \frac{1}{\lambda\varphi}|I_2 - B_2 u_{xx} - B_0 u - F u_x|^2\,dx\,dt \\ &\leq C\int_Q (I_2^2 + \lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda^3\varphi^3 u_{xx}^2)\,dx\,dt\end{aligned}$$

and

$$\begin{aligned}\int_Q \frac{1}{\lambda\varphi} u_t^2\,dx\,dt &= \int_Q \frac{1}{\lambda\varphi}|I_1 - B_1 u_x - B_3 u_{xxx} - E u|^2\,dx\,dt \\ &\leq C\int_Q (I_1^2 + \lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda\varphi u_{xxx}^2)\,dx\,dt.\end{aligned}$$

Thus we have

$$\int_Q \frac{1}{\lambda\varphi}(u_t^2 + u_{xxxx}^2)\,dx\,dt \leq C\int_Q (I_1^2 + I_2^2 + \lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda^3\varphi^3 u_{xx}^2 + \lambda\varphi u_{xxx}^2)\,dx\,dt.$$

It follows that

$$\int_Q \left(\frac{1}{\lambda\varphi} u_t^2 + \frac{1}{\lambda\varphi} u_{xxxx}^2 + \lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda^3\varphi^3 u_{xx}^2 + \lambda\varphi u_{xxx}^2\right) dx\,dt$$
$$\leq C\left(\int_Q \theta^2 f^2\,dx\,dt + \int_{Q^{\omega_0}} (\lambda^7\varphi^7 u^2 + \lambda^5\varphi^5 u_x^2 + \lambda^3\varphi^3 u_{xx}^2 + \lambda\varphi u_{xxx}^2)\,dx\,dt\right).$$

Returning u to θy, we can obtain (5.3).
Step 3. By using the same method in [28], we can eliminate the terms $\int_{Q^{\omega_0}} \lambda\varphi\theta^2 y_{xxx}^2\,dx\,dt$, $\int_{Q^{\omega_0}} \lambda^3\varphi^3\theta^2 y_{xx}^2\,dx\,dt$ and $\int_{Q^{\omega_0}} \lambda^5\varphi^5\theta^2 y_x^2\,dx\,dt$ in (5.3). Further, we have (2.1).

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